Math 4200
Friday October 16
2.4-2.5 mean value property for analytic and harmonic functions, and maximum modulus principles. But We'll begin by finishing Wednesday's notes with the introduction to the Riemann-Zeta function. The mean value property and maximum principle have many consequences, as we'll see on Monday.

Announcements:

Mean value property Let $f: A \to \mathbb{C}$ analytic, $\bar{D}(z_0; R) \subseteq A$. Then the value of f at z_0 is the average of the values of f on the concentric circle of radius R about z_0:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + R\, e^{i\theta}\right) d\theta$$

proof:

Remark harmonic functions also satsify a mean value property. How do you think you'd go about proving it?

A cool way to justify the harmonic conjugate construction in simply connected domains. ...and also to deduce that harmonic functions are infinitely differentiable.....we appealed back to multivariable calculus earlier in the course for the conjugate construction. (Harmonic conjugates show up in our proof of the mean value property for harmonic functions, at the end of Wednesday's notes.)

Theorem Let $A \subseteq \mathbb{C}$ be open and simply connected, and let $u : A \to \mathbb{R}$ be C^2 and harmonic. Then there exists a harmonic conjugate $v : A \to \mathbb{R}$, i.e. so that $f = u + i\, v$ is analytic. Furthermore, both u, v are actually C^∞, i.e. all partial derivatives exist and are continuous.

proof. If f existed, then f would be infinitely complex differentiable, and so in particular f' would be analytic...

$$f' = f_x = u_x + i\, v_x$$
$$= v_y - i\, u_y.$$

In other words,

$$g(z) = u_x - i\, u_y$$

would be analytic. Actually, CR holds for $g(z)$ defined as above just because u is harmonic and C^2, and because g has continuous first partials, so g IS analytic: Check (This was previous HW):

Since g is analytic on A and A is simply connected, g has an antiderivative $G = U + i\, V$. $G' = g$ so

$$U_x + i\, V_x = V_y - i\, U_y = u_x - i\, u_y$$

so $U_x = u_x$, $U_y = u_y$ so $U = u + C$ where C is a real constant because A is connected. Thus

$$f := G - C = u + i\, V$$

is analytic, i.e. V is a harmonic conjugate to u. Since G is infinitely complex differentiable, u, V are infinitely real differentiable.

QED.

Theorem (Maximum modulus principle). Let $A \subseteq \mathbb{C}$ be an open, connected, bounded set. Let $f: A \to \mathbb{C}$ be analytic, $f: \overline{A} \to \mathbb{C}$ continuous. Then

$$\max_{z \in \overline{A}} \{|f(z)|\} = \max_{z \in \delta A} \{|f(z)|\} := M,$$

i.e. the maximum absolute value of $f(z)$ occurs on the boundary of A. (Recall that for an open set A, the boundary $\delta A = \overline{A} \setminus A$. For general sets the boundary is the collection of points which are in the closure of the set as well as in the closure of its complement.)

Furthermore if $\exists\, z_0 \in A$ with $|f(z_0)| = M$, then f is a constant function on A.

Example: What is the maximum absolute value of $f(z) = (z-2)^2$ on the closed disk $\overline{D}(0; 2)$ and where does it occur ?

proof of maximum modulus principle: Let

$$B = \{z \in A \mid |f(z)| = M\}$$

Our goal is to show that either:

(i) $B = \emptyset$, which implies that all points in $\overline{A}$ for which $|f(z)| = M$ are on the boundary of A, as the theorem claims. And in this case there is no $z_0 \in A$ with $|f(z_0)| = M$.

OR

(ii) $B = A$. In this case $|f(z)|$ is constant. Write $f = u + i\,v$ and so we have

$$u^2 + v^2 \equiv M^2$$

If $M = 0$ then $f = 0$ on A and we are done. Otherwise $M > 0$ and taking x and y partials we get the system for each $z \in A$:

$$\begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $M \neq 0$, $(u, v) \neq (0, 0)$ at any point. Thus the determinant of the matrix is identically zero. But the determinant of the matrix is

$$u_x v_y - u_y v_x = u_x^2 + u_y^2 = v_y^2 + v_x^2.$$

Thus the gradients of u, v are identically zero on the connected open set A, so u and v are each constants on A and f is as well. This must be the case that occurs if $\exists\ z_0 \in A$ with $|f(z_0)| = M$.

Following the outline on the previous page, we have

$$B = \{ z \in A \mid |f(z)| = M \}.$$

Suppose we are not in case (i), i.e. $B \neq \varnothing$. We will show that B is open and closed in A which will imply that B must be all of A, since A is connected. Thus we are in case (ii).

Why is B closed in A?

To show B is open, let $z_0 \in B$, $\mathrm{D}(z_0, \rho) \subseteq A$. We'll show $|f(z)| = M$ $\forall\, z \in \mathrm{D}(z_0, \rho)$. Each such z in the disk is of the form $z = z_0 + r\,\mathrm{e}^{i\theta}$ with $r < \rho$. But for $0 < r < \rho$ we have the mean value property

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + r\,\mathrm{e}^{i\theta}\right) \, d\theta.$$

Use this and $|f(z_0)| = M$ to show each $\left|f\left(z_0 + r\,\mathrm{e}^{i\theta}\right)\right| = M$ as well.

This generalizes an exercise that is due today (2.4.4) where you assume a domain A was bounded by a simply connected, p.w. C^1 contour and probably use the CIF.

<u>Theorem</u>
Let $A \subseteq \mathbb{C}$ be an open, connected, bounded set. Let $f, g : A \to \mathbb{C}$ be analytic, $f, g : \overline{A} \to \mathbb{C}$ continuous. Then

$$\max_{z \in \overline{A}} \{ |f(z) - g(z)| \} = \max_{z \in \delta A} \{ |f(z) - g(z)| \}.$$

In particular, if $f = g$ on δA, then $f = g$ on all of A.

proof:

Theorem (Maximum and minimum principle for harmonic functions). Let $A \subseteq \mathbb{R}^2$ be an open, connected, bounded set. Let $u : A \to \mathbb{R}$ be harmonic and C^2, $u : \overline{A} \to \mathbb{R}$ continuous. Then

$$\max_{(x, y) \in \overline{A}} \{u(x, y)\} = \max_{(x, y) \in \delta A} \{u(x, y)\} := M,$$
$$\min_{(x, y) \in \overline{A}} \{u(x, y)\} = \min_{(x, y) \in \delta A} \{u(x, y)\} := m,$$

Furthermore if $\exists \ (x_0, y_0) \in A$ with $u(x_0, y_0) = M$ or $u(x_0, y_0) = m$, then u is a constant function on A.

Example: $u(x, y) = x^2 - y^2$ is harmonic. Where are the maximum and minimum values of u attained, on $\overline{\mathrm{D}}(0; 2)$?

proof: The maximum principle implies the minimum principle, since the minimum principle for $u(x, y)$ is equivalent to the maximum principle for $v(x, y) = -u(x, y)$. In other words, minimum values for $u(x, y)$ correspond to maximum values for $-u(x, y)$, and u is harmonic if and only if $-u$ is. So we'll focus on the maximum principle. The key tool is the mean value principle for harmonic functions: For every closed disk in A, the average value of u on the bounding circle equals the value at the center. Can you see how the proof goes, if we follow the outline of the maximum modulus principle proof?

Use: if $\mathrm{D}(z_0; \rho) \subseteq A$ then for each $0 < r < \rho$,

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(x_0 + r\cos(\theta), y_0 + r\sin(\theta)\right) \, d\theta$$

Math 4200-001
Week 8 concepts and homework
2.4-2.5
Due Friday October 23 at 11:59 p.m.

2.5 2, 5, 7, 8, 10, 15, 18.
3.1 4, 6, 7, 12